Andreev-Bashkin effect and knot solitons in an interacting mixture of a charged and a neutral superfluid with possible relevance for neutron stars

Egor Babaev
Cornell University, 620 Clark Hall, Ithaca, New York 14853-2501, USA
and Norwegian University of Science and Technology, Trondheim, N-7491 Norway
and Institute for Theoretical Physics, Uppsala University, Box 803, S-75108 Uppsala, Sweden
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We discuss a mixture of interacting neutral and charged Bose condensates, which is supposed being realized in the interior of neutron stars in the form of a coexistent neutron superfluid and protonic superconductor. We show that in this system, besides ordinary vortices of the $S^1 \to S^1$ map, the neutron condensate also allows for (meta)stable finite-length knotted solitons, which are characterized by a nontrivial Hopf invariant and in some circumstances may be stabilized by a Faddeev-Skyrme term induced by the drag effect. We also consider a helical protonic flux tube in this system and show that, in contrast, it does not induce a Faddeev-Skyrme term.

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I. INTRODUCTION

In the standard model for a neutron star its interior features superfluidity of neutron Cooper pairs and superconductivity of proton Cooper pairs (see, e.g., [1,2]). Both these condensates allow vortices of the $S^1 \to S^1$ map. Earlier it was suggested that the phenomenon of glitches in Crab and Vela pulsars is connected with vortex matter in these stars [3]. This remains a topic of intensive studies and discussions (for recent developments and citations see [4]). Besides that the standard model for a neutron star is a special system, being a mixture of interacting charged and neutral condensates which makes it also a topic of abstract academic interest [5] since such a system allows for interesting phenomena with no direct counterparts in, e.g., superconducting metals. Studies of topological defects in a mixture of charged and neutral condensates, so far, have concerned only ordinary Abrikosov-like columnar vortices (see, e.g., [6,7] and references therein). In this paper we argue that, possibly, this is not the only type of stable topological defects allowed in neutron stars. We show that due to the drag effect in a mixture of neutral and charged superfluids (Andreev-Bashkin effect) the system also allows under certain conditions stable finite-length topological defects characterized by a nontrivial Hopf invariant, more precisely a special version of knot solitons.

Finite-length topological defects characterized by a nontrivial Hopf invariant have attracted interest for a long time in condensed matter physics: earlier it was discussed in spin-1 neutral superfluids [8,9], in magnets [10], in charged and neutral two-component Bose condensates [11–13], in spin-triplet superconductors [14], and in other systems. In neutral systems finite-length closed vortices are not stable against shrinkage unless their size is stabilized by conservation of some dynamic quantity, like in the case of a propagating vortex loop. A special case is the neutral two-component system with a phase separation, where a vortex loop made up of one condensate which confines in its core circulating second condensate is stable against shrinkage [13]. Intrinsically stable topological defects characterized by a nontrivial Hopf invariant (the knot solitons) have been discussed in the Faddeev nonlinear $O(3)$ sigma model [15] where its stability is ensured by a special fourth-order derivative term

$$F_F = (\partial \vec{\mathbf{n}})^2 + \alpha(\vec{\mathbf{n}} \cdot \partial_i \vec{\mathbf{n}} \times \partial_j \vec{\mathbf{n}})^2 + \kappa(1 - \vec{\mathbf{n}} \cdot \vec{\mathbf{n}}_0)^2, \tag{1}$$

where $\vec{\mathbf{n}} = (n_1, n_2, n_3)$ is a three-component unit vector. A knot soliton (being in the simplest case a toroidal vortex loop) is a configuration where the vector $\vec{\mathbf{n}}$ resides in the core on, e.g., the south pole of the unit sphere; at infinity it reaches the north pole, while in between the core and vortex boundary it performs n rotations if one goes once around the core and m rotations if one goes once along a closed curve in the toroidal direction. The stability of knots in this model was extensively studied in numerical simulations [16]. Recently it was realized that this model is relevant for a wide class of physical systems. First, it was suggested that this model may be relevant in the infrared limit of QCD with the knot solitons being a candidate for glueballs [17]. Besides that an extended version of the Faddeev model has been derived for two-gap superconductors [11] and for triplet superconductors [14].

Below we discuss the possibility of the formation of finite length stable topological defects in a mixture of interacting charged and neutral Bose condensates.

II. MIXTURE OF INTERACTING CONDENSATES

A mixture of a charged (made up of protonic Cooper pairs) and neutral (made up of neutronic Cooper pairs) Bose condensates in the interior of neutron stars can be described in the hydrodynamic limit by the Ginzburg-Landau functional [6,7]

$$F = \frac{1}{2}\rho^{pp}\mathbf{v}_p^2 + \frac{1}{2}\rho^{nn}\mathbf{v}_n^2 + \rho^{pn}\mathbf{v}_p \cdot \mathbf{v}_n + V + \frac{\mathbf{B}^2}{8\pi}, \tag{2}$$

where $\mathbf{B}$ is the magnetic field and

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$$V=a_p|\Psi_p|^2+\frac{b_p}{2}|\Psi_p|^4+a_n|\Psi_n|^2+\frac{b_n}{2}|\Psi_n|^4+c|\Psi_p|^2|\Psi_n|^2 \tag{3}$$

is the potential term. We begin with a discussion of the simplest case of two *s*-wave condensates (so $\Psi_p=|\Psi_p|e^{i\phi_p}$ and $\Psi_n=|\Psi_n|e^{i\phi_n}$ are complex scalar fields which describe proton and neutron condensates correspondingly). In the above expression,

$$\mathbf{v}_n=(\hbar/2m_n)\nabla\phi_n \tag{4}$$

and

$$\mathbf{v}_p=(\hbar/2m_p)\nabla\phi_p-(2e/m_pc)\mathbf{A} \tag{5}$$

are superfluid velocities of neutron and proton condensates. The key feature of this system is the Andreev-Bashkin effect [18]: as a result of the interaction between two superfluids, the particle current of one of the condensates is carried by the superfluid velocity of another, so the superfluid mass current of protons and neutrons in such a system is [6,7,18]

$$\mathbf{w}_p=\rho^{pp}\mathbf{v}_p+\rho^{pn}\mathbf{v}_n,$$
$$\mathbf{w}_n=\rho^{nn}\mathbf{v}_n+\rho^{np}\mathbf{v}_p, \tag{6}$$

where $\rho^{pn}=\rho^{np}$ is the superfluid density of one of the condensates which is carried by the superfluid velocity of another. Because of the Andreev-Bashkin effect, the charged supercurrent in this system depends on the gradients of the neutron condensate [as follows from Eqs. (2),(6)]:

$$\mathbf{J}=\frac{e\hbar\rho^{pp}}{m_p^2}\left(\frac{\rho^{pn}m_p}{\rho^{pp}m_n}\nabla\phi_n+\nabla\phi_p-\frac{4e}{c\hbar}\mathbf{A}\right). \tag{7}$$

Let us discuss topological defects, allowed in Eq. (2), other than Abrikosov vortices.

III. HELICAL NEUTRON VORTEX LOOP

Let us consider a vortex loop made up of a neutron condensate with zero density of neutron Cooper pairs in its core. Let us introduce a new variable θ as follows: $\rho^{pn}m_p/\rho^{pp}m_n=\sin^2(\theta/2)$. We will consider a defect where if we go from the core center to the boundary of the flux tube in a cross section to the vortex, the variable θ grows from 0 to π. Since at the center of the vortex we have chosen that the density of the neutron condensate vanishes, then indeed there is no drag effect in the center of the flux tube and correspondingly ρ^{pn} is zero in the core. This allows one to choose the boundary condition $\sin^2(\theta/2)=0$ in the center of the vortex. Let us now impose the following configuration of ϕ_n: if we go once around the vortex core, ϕ_n changes $2\pi n$, while if we cover the vortex loop once in the toroidal direction (a closed curve along the core), ϕ_n changes $2\pi l$ with n,l being integer. This configuration corresponds to a spiral superflow of the neutron Cooper pairs in such a vortex ring. Topologically such a vortex is equivalent to the knot solitons considered in [11] and can also be characterized by a unit vector $\vec{\mathbf{e}}=(\cos\phi_n\sin\theta,\sin\phi_n\sin\theta,\cos\theta)$ with a nontrivial winding. We stress that we do not impose a nontrivial winding on ϕ_p (compare with the discussion of knot solitons in the two-gap model [11] where, in contrast, in a knot soliton the phases of both condensates must have a nontrivial winding number; however, as discussed below, the neutral-charged mixture is principally different from the system in Ref. [11]).

Indeed the nontrivial superflow of neutron Cooper pairs induces a drag current of proton Cooper pairs which in turn induces a magnetic field which can be calculated from Eq. (7):

$$\mathbf{B}=\mathrm{curl}\left[-\mathbf{J}\frac{cm_p^2}{4e^2\rho^{pp}}+\frac{c\hbar}{4e}\frac{\rho^{pn}m_p}{\rho^{pp}m_n}\nabla\phi_n\right], \tag{8}$$

which can also be written as

$$B_k=-\frac{cm_p^2}{4e^2\rho^{pp}}[\nabla_iJ_j-\nabla_jJ_i]+\frac{c\hbar}{8e}\sin\theta[\nabla_i\theta\nabla_j\phi_n-\nabla_j\theta\nabla_i\phi_n]. \tag{9}$$

This self-induced magnetic field gives the following contribution to the free energy (2):

$$F_m=\frac{\mathbf{B}^2}{8\pi}=\frac{c^2\hbar^2}{512\pi e^2}\Bigg[\frac{2m_p^2}{\hbar^2e\rho^{pp}}[\nabla_iJ_j-\nabla_jJ_i] -\sin\theta[\nabla_i\theta\nabla_j\phi_n-\nabla_j\theta\nabla_i\phi_n]\Bigg]^2, \tag{10}$$

which is a version of the Faddeev fourth-order derivative term analogous to the fourth-order derivative term in [11,14] closely related to the fourth-order derivative term in Eq. (1). The fourth-order derivative terms of this type provide stability to finite-length topological defects [15,16]. Physically, in a mixture of charged and neutral condensates this effect corresponds to the following situation: as mentioned above, the nontrivial configuration of the phase and density of neutron condensate induces a charged drag current of proton Cooper pairs which results in the configuration of the magnetic field (9). This configuration has the special feature that if the vortex shrinks, then the magnitude of the self-induced magnetic field grows. We also remark that ρ^{pp} is a measure of the background density of the proton condensate which is not required to vary to produce a knot soliton.

In the two-gap model in [11] there is competition of the fourth-order derivative term (which corresponds to self-induced magnetic field) versus a second-order derivative term and a mass term for the third component of the $O(3)$-symmetric order parameter $\vec{\mathbf{n}}$ (the third component of $\vec{\mathbf{n}}$ is related to condensate densities in [11] and thus it is massive). In contrast, in the present model in the competition there is also a contribution of the kinetic energy of the superflow of neutron Cooper pairs (which is minimized if the vortex shrinks). A necessary condition for (meta)stability of such a vortex loop is that the competition of the kinetic energy of superflows, gradients of condensate density versus the self-induced magnetic field would stabilize the vortex loop at a length scale which corresponds to the magnitude of magnetic field $|\mathbf{B}(\mathbf{x})|$ smaller than the field which could break proton Cooper pairs. We also emphasize that one of the differences with the system of two charged scalar fields in [11] is that in the present case the self-induced magnetic field comes from the drag current in the vicinity of the core while the superflow of neutron Cooper pairs is extended (not localized on a length scale shorter or equal to the penetration length like the field inducing currents in [11]). We also remark that indeed the effective action (2) is assumed as being derived from a microscopic theory in the approximation of small gradients. Indeed one can derive higher-order derivative terms from a microscopic theory but this sort of terms, in contrast to the term (10), is irrelevant for discussion of the stability of finite-length topological defects in this system. Indeed competition between second- and fourth-order derivative terms obtained in a derivative expansion would stabilize a topological defect at a characteristic length scale where all the higher-order derivative terms become of the same order of magnitude. So at such length scales the derivative expansion fails. We also would like to stress that in the present system the knot soliton is prevented against a collapse by a finite-energy barrier, in contrast to an infinite-energy barrier in the case of Faddeev's nonlinear σ model considered in mathematical physics [15]. That is, a zero in proton condensate density, outside core, may lead to an unwinding of a knot soliton since in a such a point the unit vector $\vec{\mathbf{n}}$ is ill defined and thus the Hopf map is ill defined as well. However, the modulus of the proton condensate order parameter is massive, so producing such a singularity is energetically expensive.

IV. EXAMPLE OF GENERALIZATION TO OTHER PAIRING SYMMETRIES

Let us generalize the discussion to the case of a mixture of a spin-triplet neutron condensate and a singlet proton condensate in order to show that the picture does not depend significantly on pairing symmetry. The order parameter of the spin-1 neutral condensate is $|\Psi_n(\mathbf{x})|^2\zeta_q(\mathbf{x})$ where $(q=1,0,-1)$ and ζ is a normalized spinor $\zeta^\dagger\cdot\zeta=1$. The free energy of a neutral spin-1 system is (see, e.g., [19])

$$F_t=\frac{\hbar^2}{2m_n}(\nabla|\Psi_n|)^2+\frac{\hbar^2}{2m_n}|\Psi_n|^2(\nabla\zeta)^2-\mu|\Psi_n|^2 + \frac{|\Psi_n|^4}{2}[c_0+c_2\langle\mathbf{S}\rangle^2], \tag{11}$$

where $\langle\mathbf{S}\rangle=\zeta_q^*\mathbf{S}_{qj}\zeta_j$ is the spin. Degenerate spinors are related to each other by the gauge transformation $e^{i\phi_n}$ and spin rotations $\mathcal{U}(\alpha,\beta,\tau)=e^{-iF_z\alpha}e^{-iF_y\beta}e^{-iF_z\tau}$, where (α,β,τ) are the Euler angles. Topological defects in neutral systems like this have been intensively studied (see, e.g., [8,9]). A charged counterpart of this system in the ferromagnetic state allows stable knot solitons as it was shown in [14].

Let us consider first the ferromagnetic state (which emerges when $c_2<0$) in the context of a mixture of superfluids. The energy in this case is minimized by $\langle\mathbf{S}\rangle^2=1$ and the ground-state spinor and density are [19]

$$\zeta=e^{i(\phi_n-\tau)}\left(e^{-i\alpha}\cos^2\frac{\beta}{2},\sqrt{2}\cos\frac{\beta}{2}\sin\frac{\beta}{2},e^{i\alpha}\sin^2\frac{\beta}{2}\right),$$

$$|\Psi_n|^2=\frac{1}{c_0+c_2}\mu.$$

The superfluid velocity in the ferromagnetic case is [19] $\mathbf{v}_n=(\hbar/2m_n)[\nabla(\phi_n-\tau)-\cos\beta\nabla\alpha]$. So in a mixture of a neutral ferromagnetic triplet condensate and a charged singlet condensate the equation for charged current is

$$\mathbf{J}=\frac{e\hbar}{m_p}\frac{\rho^{pp}}{m_p}\left(\frac{\rho^{pn}m_p}{\rho^{pp}m_n}[\nabla(\phi_n-\tau)-\cos\beta\nabla\alpha]+\nabla\phi_p-\frac{4e}{c\hbar}\mathbf{A}\right). \tag{12}$$

From this expression we can see that assuming, e.g., that there is no nontrivial windings in the variables α and β, the system reduces to Eq. (7) and thus allows for topological defects in the form described in the first part of the paper. We emphasze that there are no knots of this type in the charged ferromagnetic triplet system considered in [14] because in the current equation of a charged triplet superconductor, the ratio of the coefficients in front of the vector potential term and the gradient term analogous to $\nabla(\phi_n-\tau)$ does not depend on the carrier density and thus one cannot obtain a contribution analogous to the Faddeev term to the free energy by imposing a nontrivial configuration of the first gradient term in the current equation similar to Eq. (12) in the system [14]. In a charged triplet case the knot soliton may form only as a spin texture [14]. So a neutral-charged mixture with drag effect in its magnetic properties is principally different from a genuine charged system. Spin-texture knots can be formed in the present system too, as a configuration of the order parameter $\vec{\mathbf{s}}=(\cos\beta\sin\alpha,\sin\beta\sin\alpha,\cos\beta)$ characterized by a nontrivial Hopf invariant. Such a texture generates a magnetic field due the drag current induced by the superflow of the neutron Cooper pairs, which is produced by the spin texture. So, in general, there is the following nontrivial magnetic energy contribution to the free energy functional:

$$F_m^t=\frac{c^2\hbar^2}{512\pi e^2}\left[\frac{2m_p^2}{\hbar^2 e\rho^{pp}}[\nabla_iJ_j-\nabla_jJ_i] -\sin\theta[\nabla_i\theta\nabla_j\phi_n-\nabla_j\theta\nabla_i\phi_n] -\sin\beta[\nabla_i\beta\nabla_j\alpha-\nabla_j\beta\nabla_i\alpha]\right]^2. \tag{13}$$

It must be observed that the spin-texture knot soliton is structurally different from the topologically equivalent knot of the type considered in the first part of the paper. The spin-texture knot is coreless (there are no zeros of the condensate density in it). The third component of the order parameter $\vec{\mathbf{s}} = (\cos\beta\sin\alpha, \sin\beta\sin\alpha, \cos\beta)$ is massless in this case; thus, the spin-texture knot solitons in this system are energetically less expensive and have a larger characteristic size than the topologically equivalent knots in the variable $\vec{\mathbf{e}} = (\cos\phi_n\sin\theta, \sin\phi_n\sin\theta, \cos\theta)$.

Let us now consider the "polar" phase of triplet superconductors which is the case when $c_2 > 0$ in Eq. (11). The energy is minimized then by $\langle \mathbf{S}\rangle = 0$. The spinor ζ and the condensate density in the ground state are [19]

$$\zeta = e^{i\phi_n}\left(-\frac{1}{\sqrt{2}}e^{-i\alpha}\sin\beta, \cos\beta, \frac{1}{\sqrt{2}}e^{i\alpha}\sin\beta\right),$$

$$|\Psi_n|^2 = \mu/c_0.$$

The superfluid velocity in this case is (see, e.g., [19]) $\mathbf{v}_n = (\hbar/2m_n)\nabla\phi_n$ which is analogous to singlet case. Thus in the antiferromagnetic case the allowed knot solitons are equivalent to knots in a mixture of two singlet condensates considered in the first part of the paper.

We also remark that it is generally assumed that there is no proton-neutron pairing in a neutron star because of the large differences in their Fermi energies. However, certain studies argue that still such pairing can occur as a small effect. Let us consider now this situation as a theoretical example. In such a case in this system one has two charged condensates: formed by proton-proton Cooper pairs (with charge $2e$) and proton-neutron Cooper pairs (with charge e). In that case there allowed knots similar to that considered in [11] (for a discussion of columnar vortices in two-gap system see [20]).

While we cannot make at this stage any definite predictions (which would require large-scale numerical simulations), let us, however, discuss possible mechanisms of the formation of knot solitons of the type discussed above in neutron stars. As is known, ordinary vortices in superconductors form, e.g., as an energetically preferred state in an external magnetic field. Indeed it is not the only possible mechanism of the creation of topological defects. For instance many defects in liquid helium are created in a laboratory without rotation, by various techniques including neutron irradiation. Besides that a symmetry breaking phase transition is accompanied by the creation of topological defects. In the case of the ordinary Abelian Higgs model, created during the transition, vortex loops shrink, while the knot solitons should remain stable. Another less common mechanism of the formation of vortices is the spontaneous vortex state [21] which emerges when in a system superconductivity and magnetism coexist. Since a vortex carries a magnetic field, it may have a negative contribution to the free energy functional via Zeeman-like coupling terms. So for such systems it is energetically preferred to form a vortex lattice even without applied external fields [21]. A similar mechanism may work in neutron stars if in the presence of a spin-1 superfluid, there is a direct coupling of spins of Cooper pairs to the magnetic field. In general an effective action of a triplet superfluid features a Zeeman-like term which is a direct coupling of spin $\mathbf{S}$ to magnetic field $\mathbf{B}$:

$$F_Z = -\eta\mathbf{S}\cdot\mathbf{B}. \tag{14}$$

Indeed the existence of such a term could result in a range of parameters where knots would have a finite negative energy if the spins of neutron Cooper pairs in the knot soliton are aligned along the self-induced magnetic field. A definite answer to this question is, however, a complicated problem. Considering, e.g., the case of a mixture of two charged condensates and a neutral triplet condensate with a Zeeman term, we may construct a knot in the charged doublet with spins of the neutral condensate aligned along the self-induced field, in order to have a negative Zeeman term. Then we have a complicated situation of competition of many terms including second-order gradient terms of spin variables, second-order gradient terms of charged condensates, and fourth-order derivative terms including those which come from Andreev-Bashkin drag currents. This appears to be a particularly interesting problem for numerical simulations. Since in a knot the magnetic field grows if the knot shrinks, it could be that in a such a system the formation of a dense ensemble of knot solitons is energetically preferred over a spontaneous vortex state of Abrikosov vortices. A definite answer to this question may, however, be only obtained in a large-scale numerical simulation. Thus, if an ensemble of knot solitions is formed in a neutron star, then one of the apparent consequences would be its interaction with ordinary columnar neutron vortices; then, apparently in such a case knot solitons would disturb a regular lattice of neutron vortices.

V. HELICAL PROTONIC FLUX TUBE FORMED AROUND A NEUTRON VORTEX

Above we considered knot solitons which appear due to nontrivial helical windings of the neutron condensate phase. In principle there is a theoretical mechanism which would allow the formation of helical vortex loops of proton condensate. Let us show, however, that a helical protonic vortex loop is not a knot soliton and it is not stable. Here we stress that most recent studies [4] indicate type-I behavior of proton condensates. Let us now, however, consider the model [22]. In that model a neutron star possesses a lattice of uniform neutron vortices and a complicated structure of sparse entangled proton flux tubes (see Fig. 3 in [22]). In the dynamical processes discussed in [22] one should expect that entangled proton flux tubes may dynamically form rings around columnar neutron vortices as shown on Fig. 1. Let us now consider such a ring. The charge current in such a ring is given by Eq. (7). When we go around such a flux tube, the protonic phase ϕ_p changes 2π; however, there is also a current along such a vortex due to the drag effect by superfluid neutron Cooper pairs which is characterized by a nontrivial phase winding of ϕ_n which changes 2π when we cover the flux tube once in the toroidal direction. This results in a spiral net charge current in such a vortex loop resembling that of a knot solition considered in the first part of the paper. Let us show, however, that such a vortex loop is not stable:

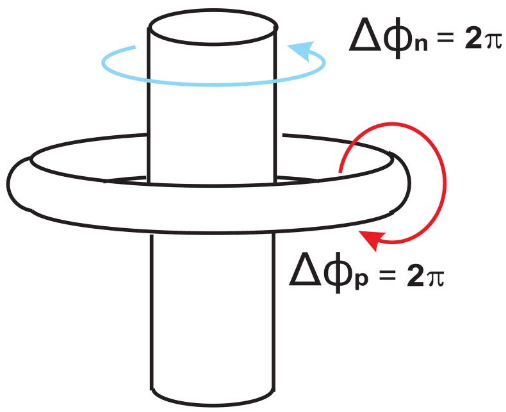

FIG. 1. Protonic flux tube ring around a neutron vortex. As a result of drag by neutron Cooper pairs, the resulting charge current in the protonic flux tube is helical.

The magnetic field in such a helical fluxtube is given by

$$\mathbf{B} = \frac{c\hbar}{4e}\,\mathrm{curl}\left[-\mathbf{J}\frac{m_p^2}{e\hbar\rho^{pp}} - \nabla\phi_p - \frac{\rho^{pn}m_p}{\rho^{pp}m_n}\nabla\phi_n\right]. \qquad (15)$$

In such a configuration, in spite of helical net charge current, the individual phase configurations of ϕ_p and ϕ_n are not helical; besides that, the ratio of the vector potential term to gradient term for ϕ_p is constant. Thus such a helical superflow does not result in a self-induced Faddeev-Skyrme-like term, which, if present, would significantly affect the considerations in [4,22].

VI. CONCLUSION

In conclusion we studied topological defects other than Abrikosov vortices in an interacting mixture of neutral and charged condensates. Such a system is believed to be realized in the interior of neutron stars. We have shown that due to the Andreev-Bashkin effect the system possesses a large variety of knot solitons of different nature than the knot solitons in the systems studied before. We also suggested that due to the Zeeman coupling term, there could be the theoretical possibility of an exotic inhomogeneous ground state in this system: a spontaneous formation of a dense ensemble of knot solitons.

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